INVERSE TRIGONOMETRIC FUNCTIONS

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Introduction

- Periodicity
- Odd and Even functions
- Degree and Radian Measures
- Domain and Range of trigonometric functions
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- Restricting domains of trigonometric functions

Objectives

- Definitions of inverse trigonometric functions.
- Graphing inverse trigonometric functions
- Evaluate inverse trigonometric functions.
- Use trigonometric equations and inverse trigonometric functions to solve problems.

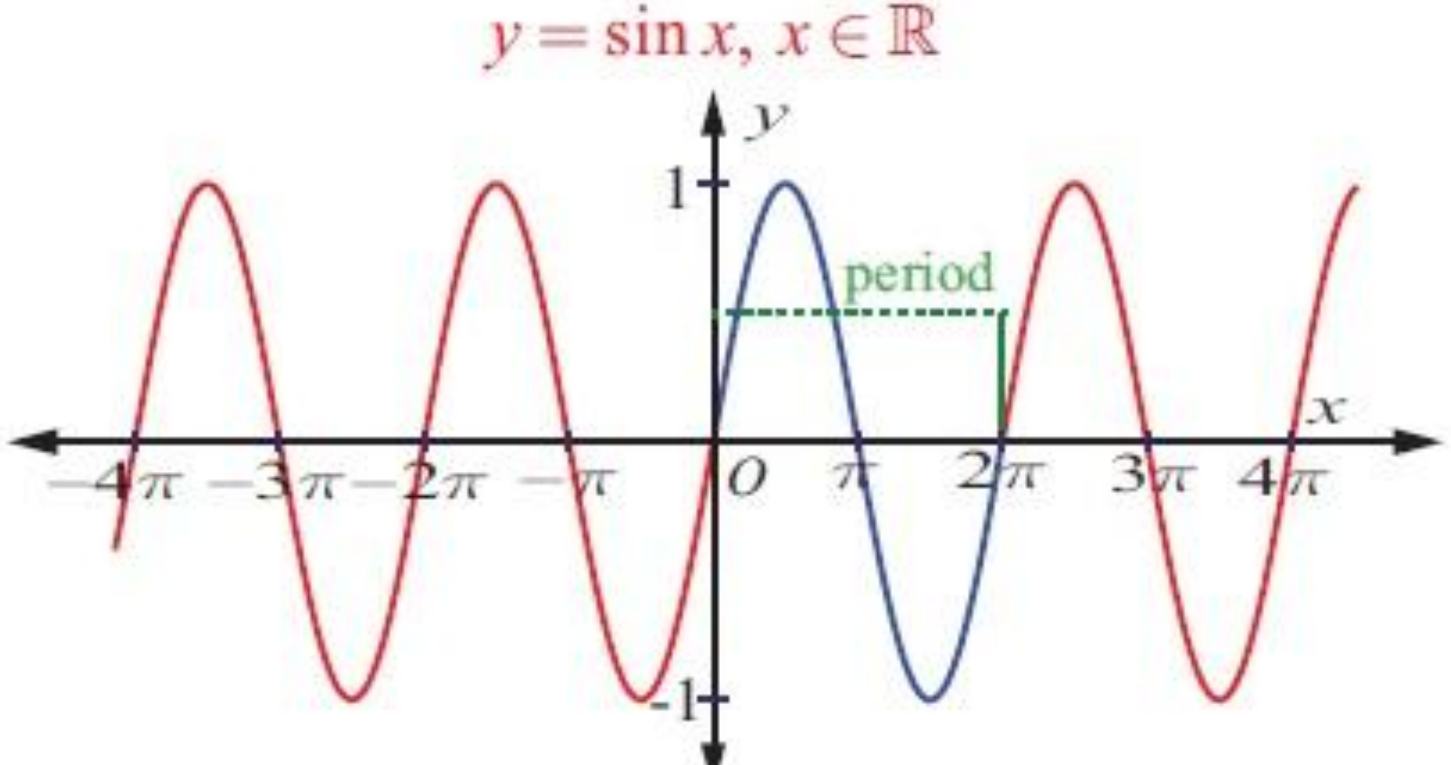
Inverse Trigonometric Functions

- We know how to evaluate trigonometric functions for a given angle.
- Also, we can find the measure of an angle given the values of trigonometric functions by using an inverse trigonometric function.

What is the domain and range of the Sine function

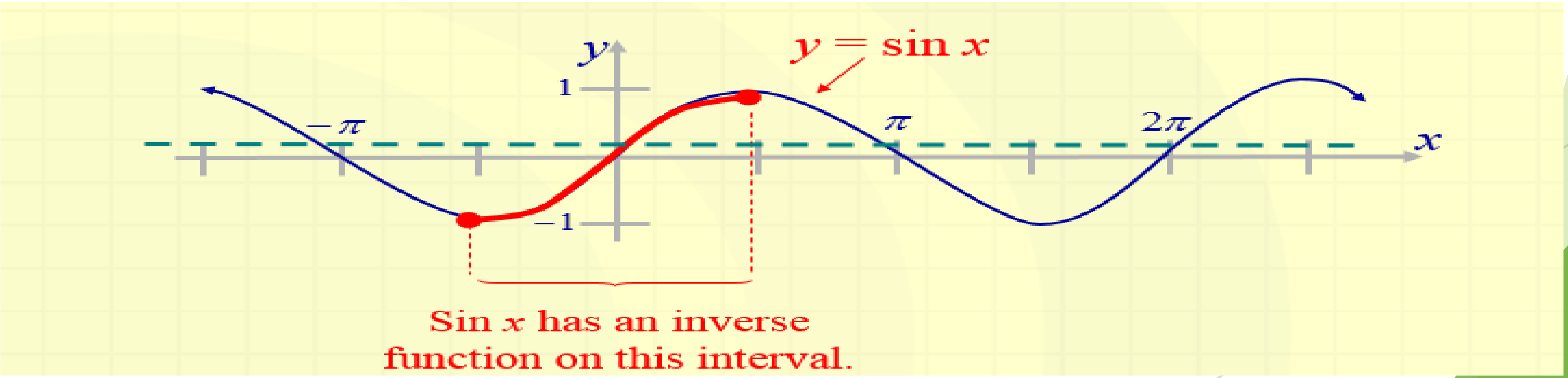
Domain: *All real numbers*

Range: [-1, 1]



Inverse Sine Function

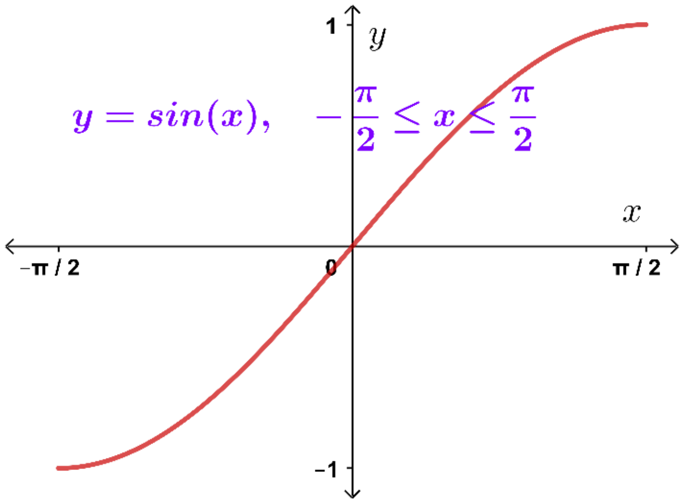
- Recall that for a function to have an inverse, it must be a one-to-one function or pass the horizontal line test.
- $f(x)=\sin x$ does not pass the horizontal line Test and the domain must be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,but still the range is whole [-1, 1], to find its inverse

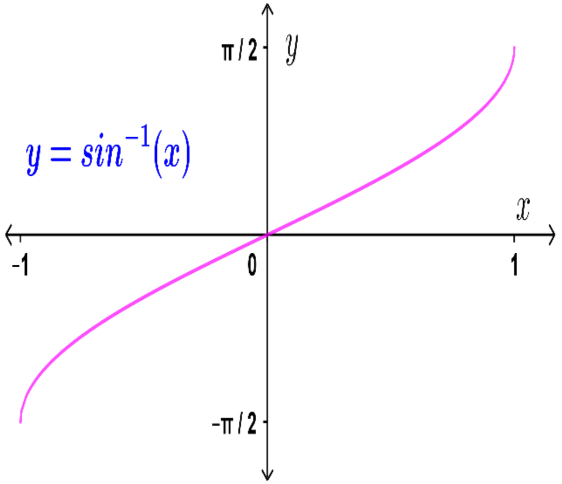


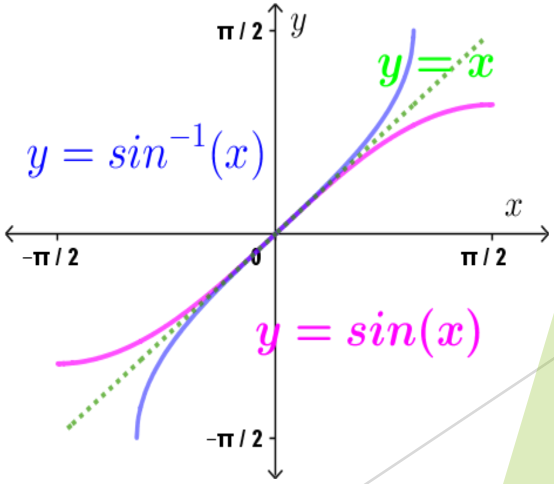
What is the domain and range of the Inverse of the Sine function

The inverse sine function : Domain [-1, 1]

The Range is not all real numbers, but $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$







- The graph of $y = \sin^{-1}x$ can be obtained by reflecting the graph of

  $y = \sin x$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ about the line $y = x$.

- Its x-intercept is 0 and y-intercept is 0.

- $y = \sin^{-1}x$ is an odd function, since the graph is symmetric with respect

  to the origin.

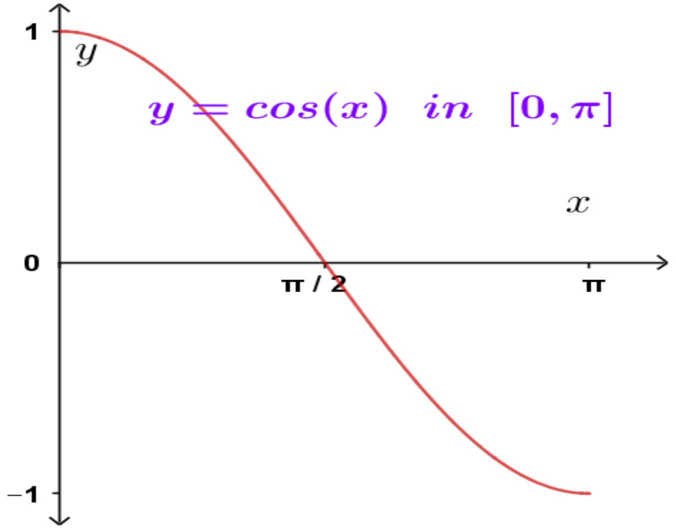
- The graph of the function $y = \sin^{-1}x$ can also be obtained from the

  graph y = sinx, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ by interchanging x and y axes.

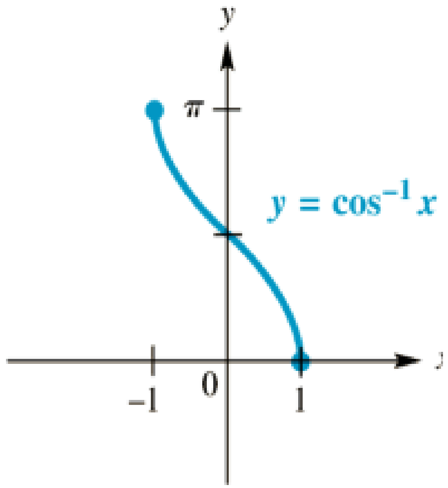
What is the domain and range of the Inverse of the Cosine function

Domain: [-1,1]

Range: $[0, \pi]$



x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0

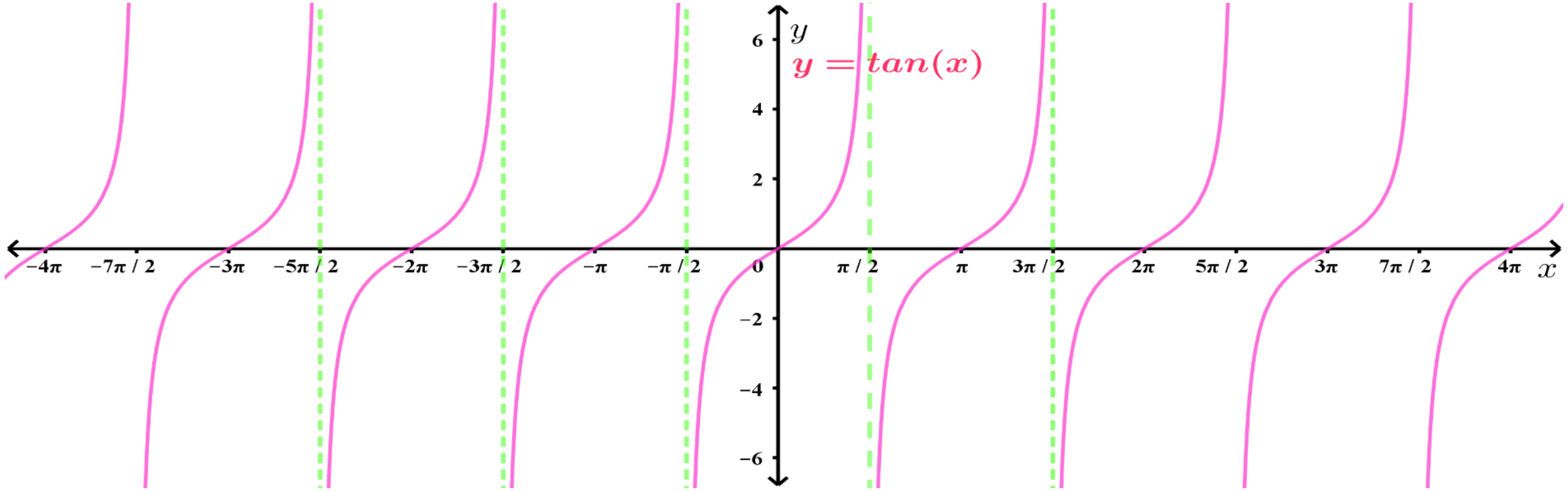


- For the function $y = \cos^{-1} x$, the *x*-intercept is 1 and the *y*-intercept is $\frac{\pi}{2}$
- The graph is not symmetric with respect to either origin or y-axis, So $y = \cos^{-1} x$ is neither even nor odd function.
- The graph of the function $y = \cos^{-1} x$ can also be obtained from the graph of $y = \cos x,\ x \in [0,\ \pi]$ by interchaning *x* and *y* axes

What is the domain and range of the Tangent function

Domain: All real numbers except $\frac{\pi}{2} + n\pi$ Where *n* is a integer

Range: All real numbers

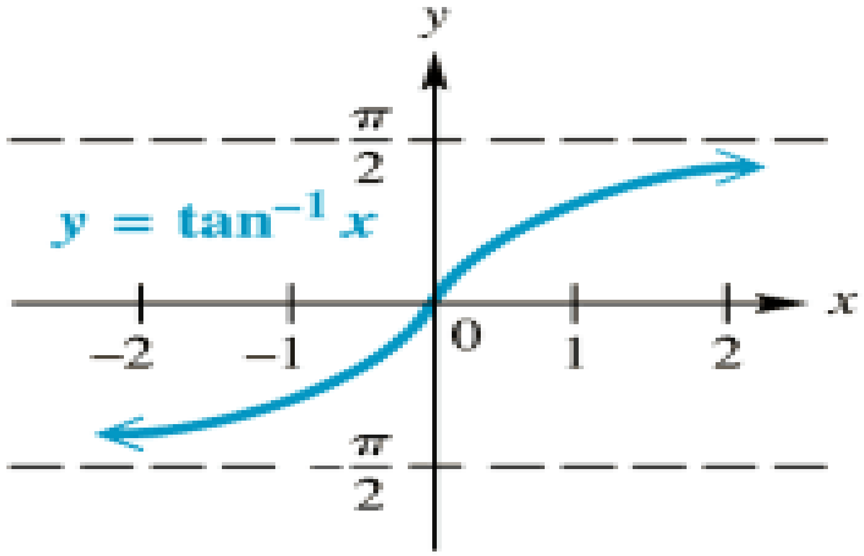


What is the domain and range of Inverse of Tangent function

Domain: All real numbers

Range: $\left[-\frac{\pi}{2}, \ \frac{\pi}{2}\right]$

x	y
-1	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$
0	0
$\frac{\sqrt{3}}{3}$	$\frac{\pi}{6}$
1	$\frac{\pi}{4}$



The arcsine or the inverse function of the sine.

What is the angle that has a sine equal to a given number $\frac{1}{\sqrt{2}}$

Since,

$$\arcsin\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

4.2.1 Domain and Range of trigonometric functions

The **domain** and **range** of trigonometric functions are given in the following table.

Trigonometric function	$\sin x$	$\cos x$	$\tan x$	$\operatorname{cosec} x$	$\sec x$	$\cot x$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$	$\mathbb{R} \setminus \{n\pi, n \in \mathbb{Z}\}$	$\mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$	$\mathbb{R} \setminus \{n\pi, n \in \mathbb{Z}\}$
Range	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$	$\mathbb{R} \setminus (-1, 1)$	$\mathbb{R} \setminus (-1, 1)$	$\mathbb{R}$

Domain and range Inverse Trigonometry Function

function	Domain	Range (principal value)
1. $y=\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2. $y=\cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3. $y=\tan^{-1} x$	$\mathbb{R}$	$-\frac{\pi}{2} < y \frac{\pi}{2}$
4. $y=\operatorname{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} < y < \frac{\pi}{2}, y \in 0$
5. $y=\sec^{-1} x$	$x \ge 1$ or $x \le -1$	$0 < y \le \pi, y \ne \frac{\pi}{2}$
6. $y=\cot^{-1} x$	$\mathbb{R}$	$0 < y < \pi$

Principal value of inverse trigonometric functions

- The value of an inverse trigonometric function at a point *x* of its domain which lies in the range of principal branch is the principal value of the inverse trigonometric functions at that point *x*. For instance, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *is* $\frac{\pi}{6}$, *since* $\frac{\pi}{6} \in [0,\ \pi]$

The principal value of inverse trigonometry function

Function	Principal Domain	Range
sine	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$	$[-1,1]$
cosine	$[0, \pi]$	$[-1,1]$
tangent	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$	$\mathbb{R}$
cosecant	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$	$\mathbb{R} \setminus (-1, 1)$
secant	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$	$\mathbb{R} / (-1,1)$
cotangent	$(0, \pi)$	$\mathbb{R}$

Inverse Function	Domain	Range of Principal value branch
$\sin^{-1}$	$[-1,1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1,1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
cosec^{-1}	$R \setminus (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$
$\sec^{-1}$	$\mathbb{R} \setminus (-1,1)$	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$
$\cot^{-1}$	$\mathbb{R}$	$(0, \pi)$

- Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Solution

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$. Then $\sin y = \frac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence, we must find $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \frac{1}{\sqrt{2}}$. Clearly, $y = \frac{\pi}{4}$.

Thus, the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

Properties of inverse trigonometric functions

- We shall investigate some properties of inverse trigonometric functions. The properties to be discussed are valid within the principal value branches of the corresponding inverse trigonometrical functions and where they are defined. Some of these properties are not valid for all values of x in the domain of inverse trigonometric functions

Property-I

(i) $\sin^{-1}(\sin\theta) = \theta$, if , $\in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, if $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

(v) $\sec^{-1}(\sec\theta) = \theta$, if $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot\theta) = \theta$, if $\theta \in (0, \pi)$

Property-II

(i) $\sin(\sin^{-1} x) = x$, if $x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$, if $x \in \mathbb{R}$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, if $x \in \mathbb{R} \setminus (-1, 1)$

(v) $\sec(\sec^{-1} x) = x$, if $x \in \mathbb{R} \setminus (-1, 1)$

(vi) $\cot(\cot^{-1} x) = x$, if $x \in \mathbb{R}$

Property-III (Reciprocal inverse identities)

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec} x$, if $x \in \mathbb{R} \setminus (-1, 1)$.

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec x$, if $x \in \mathbb{R} \setminus (-1, 1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & \text{if } x > 0 \\ -\text{À} + \cot^{-1} x & \text{if } x < 0. \end{cases}$

Property-IV(Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, if $x \in [-1, 1]$.

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, if $x \in \mathbb{R}$.

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$.

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, if $x \in [-1, 1]$.

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$.

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, if $x \in \mathbb{R}$.

Property-V (cofunction inverse identities)

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},\ x \in [-1, 1]$.

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},\ x \in \mathbb{R}$.

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2},\ x \in \mathbb{R} \setminus (-1, 1)$ or $|x| \geq 1$.

Property-VI

(i) $\sin^{-1}x+\sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$, where either $x^2+y^2\le 1$ or $xy<0$.

(ii) $\sin^{-1}x-\sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$,, where either $x^2+y^2\le 1$ or $xy>0$.

(iii) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x+y\ge 0$.

(iv) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left[xy+\sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x''\ y$.

(v) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy<1$.

(vi) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$ if $xy>-1$.

Property-VII

(i) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), |x|<1$

(ii) $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), x\ge 0$

(iii) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right), |x|\le 1,\ \ |x|\le 1$

Property-VIII

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$ if $|x|\le\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\cos^{-1}x$ if $\frac{1}{\sqrt{2}}\le x\le 1$.

Property-IX

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$ if $0 \le x \le 1$. (ii) $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$ if $-1 \le x < 0$.

(ii) $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ if $-1 < x < 1$. (iv) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$ if $0 \le x \le 1$.

(v) $\cos^{-1}x = \text{À} - \sin^{-1}\sqrt{1-x^2}$ if $-1 \le x < 0$. (v) $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$ if $x > 0$

.

Property-X

(i) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$. (ii) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\dfrac{1}{2}, 1\right]$.

Find the maximum and minimum values of $\sin^{-1}x + 2\cos^{-1}x$.

Solution: $\sin^{-1}x + 2\cos^{-1}x = \sin^{-1}x + \cos^{-1}x + \cos^{-1}x = \frac{\pi}{2} + \cos^{-1}x$

We know that $0 \le \cos^{-1}x \le \pi$

Thus, $\frac{\pi}{2} + 0 \le \left(\cos^{-1}x + \frac{\pi}{2}\right) \le \pi + \frac{\pi}{2}$

Hence, $\frac{\pi}{2} \le \sin^{-1}x + 2\cos^{-1}x \le \frac{3\pi}{2}$

Thus, the minimum value is $\frac{\pi}{2}$ and the maximum value is $\frac{3\pi}{2}$.